GENERALIZED MODELING AND ANALYTICAL DESCRIPTION OF META-CONSTANT PHENOMENA IN HEAT CONDUCTION AND IN HEAT AND MASS TRANSFER

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The derivation is shown of generalized quantities called integroforms, with the aid of which: a) variable coefficients are eliminated from differential equations and boundary conditions; b) one may derive the integral invariants of reciprocal similarity (or identity) transformations "as a whole" for meta-constant phenomena of a given class.

Methods of quantitative description of phenomena associated with constancy of physical coefficients $\omega =$ = const ("constant phenomena" [6]), are comparatively well developed. Here we may relate both the methods of classical mathematics, for example the theory of differential equations in partial derivatives, and also methods of the theory of classical similarity and simulation [11]. The above methods are usually not applicable [1–10] to description of phenomena associated with variation of coefficients $\omega \neq$ const ("meta-constant phenomena" [6, 10]). No general theory exists for solution of the above equations with $\omega \neq$ const.

On the basis of the ideas underlying differential and integral calculus and differential geometry, it has proved possible to devise a method of finding a system U_Z of integral forms (the method of Z integroforms). Representation of the objects of system U_Z by a "secondary information" system, replacing the "primary information" system U_q (see, for example, [10]), leads to an appreciable generalization of the description of meta-constant objects [1–10]. The integroforms Z are synthesized through "analysis of small dimensions" [5–10], followed by transition to analysis of dimensions which are "integral as a whole" [1–10]. In quantitative description of meta-constant objects the integroforms can serve to achieve two purposes: 1) derivation of the invariants B = invar (dimensionless integroforms) of generalized modeling; 2) elimination of the variable coefficients $\omega \neq$ const from the differential equations and the boundary conditions [1–10].

We give some examples below to illustrate the application of the method of integroforms to a generalized description of meta-constant phenomena of heat conduction and of heat and mass transfer.

Meta-constant phenomena of heat conduction in a solid are described by the well-known Fourier equation

$$C_V \frac{\partial T}{\partial t} = \operatorname{div}(\lambda \operatorname{grad} T), \quad (1)$$

in which we assume that

$$[C_V] = [Q/l^3T],\ [\lambda] = [Q/ltT],\ C_V = \rho l,\ a = \lambda/C_V,$$
$$C_V = f\{T(x,\ y,\ z;\ t)\} = f_1(x,\ y,\ z;\ t) \neq \text{const},$$
$$\lambda = \varphi\{T(x,\ y,\ z; t)\} = \varphi_1(x,\ y,\ z;\ t) \neq \text{const}, \quad (2)$$

where Q is the amount of heat; C_V is the volume specific heat.

Analysis of "local similarity" [5–10] leads to derivation of the invariant b_{Fo} for small scale similarity:

$$b_{Fo} = \frac{\{d(\lambda dT)\}(dt)}{C_V (dx_p)^2 (dT)} = \frac{\{d(\lambda dT)\}(adt)}{\{dx_p\}^2 (\lambda dT)}, \quad (3)$$

or

$$b_{Fo} = \frac{\{d(dX_T)\}(dX_t)}{(dx_p)^2 (dX_T)} = \text{invar}, \quad (4)$$

where on the small scale we make the substitution

$$dX_T = \lambda dT, \quad dX_t = adt. \quad (5)$$

Hence, going on to examination "over-all" [1–10], the phenomena of meta-constant heat conduction may be described by a system of quantities of only the following three types:

$$U_X = X_T \cup X_t \cup \bigcup_{p=1}^{p=3} x_p, \quad (6)$$

where the secondary information variables are retained, these being the integroforms

$$X_T = \int_{(T)} \lambda\, dT_1, \quad (7)$$

$$X_t = \int_{(t)} a dt_1. \quad (8)$$

Turning now to description of phenomena by differential equations, we see that, using (6), we may replace (1) by the equation

$$\partial X_T / \partial X_t = \Delta X_T(x,\ y,\ z;\ X_t), \quad (9)$$

which is amenable to solution by known methods.

An important point is that the theory of integroforms gives a general method of finding the integroforms, such as, for example, X_T, X_t, etc. [4–10].

The inverse transition from the system U_X to the primary information system, for example

$$U_q = T \cup C_V \cup \lambda \cup t \cup \bigcup_{p=1}^{p=3} x_p \quad (10)$$

consists of transformations following immediately from the dX expressions. For example, according to (5), we have

$$t - t_0 = \int_{X_t(t_0)}^{X_t(t,\ t_0)} a^{-1} dX_{t_1}, \quad T - T_0 = \int_{X_T(T_0)}^{X_T(T,\ T_0)} \lambda^{-1} dX_{T1}. \quad (11)$$

The result of (3) and (4), which is the original in the transformation of (1) to (9), is also a basic and generalized modeling of heat conduction phenomena.

In the special case when ω = const, we have, from (1), the well-known invariant of classical similarity

$$\mathrm{Fo} = atl^{-2} = \text{invar}. \tag{12}$$

With $\omega \neq$ const in case (2), we find, from (4) or (9) the integral invariant [4, 6]

$$B_{\mathrm{Fo}} = X_t l^{-2} = \text{invar}. \tag{13}$$

The solution of the heat conduction problem in the example given will have the form

$$X_T \equiv X_T(B_{\mathrm{Fo}} \cup B_F), \tag{14}$$

where the B_F must be found from the boundary conditions.

If

$$\omega \equiv \omega\left(T \cup t \cup \bigcup_1^3 x_p\right), \tag{15}$$

we obtain the B_{Fo}, for example in the form [6]

$$B_{\mathrm{Fo}} = \int\limits_{(t)} \left\{ \int\limits_{(l)} \left| \int\limits_{(T)} aT_1^{-1} dT_1 \right|^{-1/2} dl_1 \right\}^{-2} d\tau = \text{invar}. \tag{16}$$

When $\omega \equiv \omega(x, y, z; t)$ we obtain

$$B_{\mathrm{Fo}} = \int\limits_{(t)} \left\{ \int\limits_{(l)} |a|^{-1/2} dl_1 \right\}^{-2} d\tau = \text{invar}. \tag{17}$$

Continuing the transformation given in [15], in the case

$$a \equiv a(x, y, z; t) = a_x(x, y, z)\, a_t(t) \tag{18}$$

we obtain

$$B_{\mathrm{Fo}} = \int\limits_{(t)} a_t(t_1)\, dt_1 \left\{ \int\limits_{(x_p)} a_{x_1}^{-1/2}(x_{p_1})\, dx_{p_1} \right\}^{-2} = \text{invar}. \tag{19}$$

As an example of solution of (9) we shall examine the one-dimensional problem of propagation of heat in a plane-parallel slab:

$$C_V \frac{\partial T}{\partial t} = \frac{\partial}{\partial x}\left(\lambda(T) \frac{\partial T}{\partial x}\right); \quad T(0, x) \equiv T_0(x);$$

$$-\lambda(T) \left.\frac{\partial T}{\partial x}\right|_{x=0} = f(t), \quad -\lambda(T) \left.\frac{\partial T}{\partial x}\right|_{x=h} = 0. \tag{20}$$

According to (6)–(8), we reduce the given system to the generalized form:

$$\frac{\partial X_T}{\partial X_t} = \frac{\partial^2 X_T}{\partial x^2}; \quad X_T(0, x) = X_{T_0}(x);$$

$$-\left.\frac{\partial X_T}{\partial x}\right|_{x=0} = \psi(X_t), \quad -\left.\frac{\partial X_T}{\partial x}\right|_{x=h} = 0. \tag{21}$$

Introducing the argument $X = B_{\mathrm{Fo}}^{-1/2} = xX^{-1/2}$, we have the ordinary differential equation

$$\frac{d^2 X_T}{dX_t^2} + \frac{X}{2} \frac{dX_T}{dX} = 0, \tag{22}$$

whose solution, expanding (14), is

$$X_T = 2u_0 \int\limits_{X_0}^{X} \exp[-(X_1/2)^2]\, dX_1/2, \quad u_0 = \left.\frac{dX_T}{dX}\right|_0 = \text{const}, \tag{23}$$

where u_0 is determined by the boundary conditions.

In a one-dimensional diffusion process, described by the analogous system

$$\frac{\partial C}{\partial t} = \frac{\partial}{\partial x}\left\{D(C) \frac{\partial C}{\partial x}\right\}, \quad C(0, x) = \varphi(x),$$

$$-D(C) \left.\frac{\partial C}{\partial x}\right|_{x=0} = f(t), \quad -\left.\frac{\partial C}{\partial x}\right|_{x=h} = 0, \tag{24}$$

replacement by the generalized system

$$\frac{\partial X_C}{\partial X_t} = \frac{\partial^2 X_C}{\partial x^2}, \quad X_C(0, x) = X_{C_0}(x),$$

$$-\left.\frac{\partial X_C}{\partial x}\right|_{x=0} = \psi(X_t), \quad -\left.\frac{\partial X_C}{\partial x}\right|_{x=h} = 0, \tag{25}$$

$$X_C = \int\limits_{(C)} D\, dC_1, \quad X_t = \int\limits_{(t)} D\, dt_1,$$

yields the solution similarly (D is the diffusion coefficient).

To describe a heat and mass transfer process in a capillary-porous substance we have the well-known Lykov system of equations, which, with some changes, may be written in the form [17]

$$\frac{\partial(\rho c T)}{\partial t} = \mathrm{div}(\lambda\, \mathrm{grad} T) + E\zeta \frac{\partial(c_d \rho \Theta)}{\partial t}, \tag{26}$$

$$\frac{\partial(\rho c_d \Theta)}{\partial t} = \mathrm{div}(\lambda_d \delta\, \mathrm{grad}\, T + \lambda_d\, \mathrm{grad}\, \Theta). \tag{27}$$

We shall take the initial conditions, as in [17], for example, to be

$$T(x, y, z; 0) = T_0 = \text{const},$$

$$\Theta(x, y, z; 0) = \Theta_0 = \text{const}, \tag{28}$$

and the boundary conditions at the point χ_F on the boundary FL of region L, in which the phenomenon occurs, as in [17], for example, to be

$$\lambda_F \frac{\partial T(\chi_F, t)}{\partial n} + \lambda_F \{T_m - T(\chi_F, t)\} -$$

$$-(1 - E_F)\zeta_F \alpha_{d_F} \rho \{\Theta(\chi_F, t) - \Theta_{m_F}\} = 0, \tag{29}$$

$$\lambda_{d_F} \frac{\partial \Theta(\chi_F, t)}{\partial n} + \lambda_d \delta \frac{\partial T(\chi_F, t)}{\partial n} +$$

$$+ \alpha_{d_F} \{\Theta(\chi_F, t) - \Theta_{m_F}\} = 0. \tag{30}$$

In the one-dimensional case, for example that of a plane-parallel slab, the gradients of the functions in (26) and (27) are replaced by partial derivatives with respect to x; in (29) and (30) the derivatives with respect to the normal are replaced by derivatives with respect to x, while the points χ_F are replaced directly by the coordinate x = 0 on one plane, and by $x = l_0$ = const on the other.

Analysis of the similarity of such phenomena, both with $\omega \neq$ const, and with ω = const, leads to four invariants for the internal points $\chi_L \in L$ and to eight invariants at the points χ_F on the boundary planes. In all here there are 4 + 8, i.e., 12 invariants (criteria) of similarity. As may be seen, the system (26)-(30) is complicated, not only for finding an analytical solution, but also for analysis of similarity and of modeling. We shall therefore replace this system by the symmetrical generalized system

$$\partial X_{1T}/\partial t = -\operatorname{div} q_{T\Theta}(x, y, z; t), \tag{31}$$

$$\partial X_{1\Theta}/\partial t = -\operatorname{div} q_{\Theta T}(x, y, z; t), \tag{32}$$

in which the operator div applies to the vector fluxes

$$-q_{T\Theta} = \operatorname{grad} X_{4T\Theta}, \quad -q_{\Theta T} = \operatorname{grad} X_{\Theta T}, \tag{33}$$

where $q_{T\Theta}$ is the heat flux, $[q_{T\Theta}] = [X/m^2 \cdot sec]$, $q_{\Theta T}$ is the mass flux, and $[q_{\Theta T}] = kg/m_2 \cdot sec$.

Therefore,

$$\partial X_{1\Theta}/\partial t = \Delta X_{\Theta T}(x, y, z; t), \tag{34}$$

$$\partial X_{1T}/\partial t = \Delta X_{4T\Theta}(x, y, z; t), \tag{35}$$

where the generalized variables, integroforms of X, are

$$X_{4T\Theta} = X_{2T} + X_{3T} + X_{3\Theta}, \quad X_{\Theta T} = X_{T\delta} + X_{2\Theta},$$

$$X_{1T} = \int_{(H)} d(C_V T), \quad X_{1\Theta} = \int_{(M)} d(C_d \rho\Theta), \quad X_{2T} = \int_{(T)} \lambda\, dT_1,$$

$$X_{3T} = \int_{(T)} (\lambda_d E \zeta)\, \delta\, dT_1, \quad X_{3\Theta} = \int_{(\Theta)} \lambda_d E \zeta\, d\Theta_1,$$

$$X_{T\delta} = \int_{(T)} \lambda_d \delta\, dT_1, \quad X_{2\Theta} = \int_{(\Theta)} \lambda_d\, d\Theta_1. \tag{36}$$

Knowing (33), we may also formulate the generalized symmetrical boundary conditions, of the second kind, for example

$$-q_{T\Theta}(\chi_F, t) = \partial X_{4T\Theta_F}/\partial n, \quad -q_{\Theta T}(\chi_F, t) = \partial X_{\Theta T_F}/\partial n \tag{37}$$

or of the third kind

$$-\frac{\partial X_{4T\Theta_F}}{\partial n} = \alpha_F (X_{4T\Theta_m} - X_{4T\Theta_F}),$$
$$-\frac{\partial X_{\Theta T_F}}{\partial n} = \alpha_{d_F} (X_{\Theta T_F} - X_{\Theta T_m}). \tag{38}$$

In one-dimensional problems, (34) and (35) are transformed to the form

$$\frac{\partial X_{1T}}{\partial t} = \frac{\partial^2 X_{4T\Theta}}{\partial x^2}, \quad \frac{\partial X_{1\Theta}}{\partial t} = \frac{\partial^2 X_{\Theta T}}{\partial x^2}, \tag{39}$$

while in the boundary conditions (37) and (38), the derivatives with respect to the normal are replaced by derivatives with respect to x, and the points χ_F are replaced by coordinate x.

Analysis of the similarity of the meta-constant phenomena under examination, which are described by the system (34)-(37) or (34)-(36) and (38), leads to the invariants B = invar of generalized modeling, the number of which is now diminished by a factor of two relative to the primary information system (28)-(30). Thus, in lieu of the four invariants for the internal points $\chi_L \in L$ we now have only two invariants, B_{Fo} and B_{Fi}, which in the most general case have the form

$$B_{Fo} = \int_{(t)} \Big\{ \int_{(x_p)} dx_{p_1} \int_{(x_{p_1})} \times \Big\{ \int_{(X_{4T\Theta})} X_{1T}^{-1} dX_{4T\Theta} \Big\}^{-1} dx_{p_2} \Big\}^{-1} d\tau = \text{invar}, \tag{40}$$

$$B_{Fi} = \int_{(t)} \Big\{ \int_{(x_p)} dx_{p_1} \int_{(x_{p_1})} \times \Big\{ \int_{(X_{\Theta T})} X_{1\Theta}^{-1} dX_{\Theta T} \Big\}^{-1} dx_{p_2} \Big\} d\tau = \text{invar}. \tag{41}$$

Noting that $X_{1T} = H - H_0$, $X_{1\Theta} = M - M_0$, H_0 = const, M_0 = const, and assuming that E, ζ depend weakly on $M = \rho c_d \Theta$, we obtain, after transformations

$$B_{Fo} = \int_{(t)} \Big\{ \int_{(x_p)} \Big| \int_{(T)} \Psi\, dT_1 \Big|^{-1/2} dx_{p_1} \Big\}^{-2} d\tau = \text{invar}, \tag{42}$$

$$B_{Fi} = \int_{(t)} \Big\{ \int_{(x_p)} \Big| \int_{(\Theta)} \lambda_d (M - M_0)^{-1} d\Theta_1 \Big|^{-1/2} dx_{p_1} \Big\}^{-2} d\tau = \text{invar}, \tag{43}$$

where $\Psi = (\lambda + 2\lambda_d E\xi)(H - H_0)^{-1}$. In the special case, when ω does not depend on x, y, z, t, we obtain, for example

$$B_{Fo} = \frac{t}{l^2} \int_{(T)} \Psi\, dT_1 = \text{invar}, \tag{44}$$

$$B_{Fi} = \frac{t}{l^2} \int_{(\Theta)} \lambda_d (M - M_0)^{-1} d\Theta_1 = \text{invar}. \tag{45}$$

If ω depends only on x, y, z, t, we obtain, for example,

$$B_{Fo} = \int_{(t)} \Big\{ \int_{(x_p)} \Big| a + a_d \frac{c_d}{c} E\zeta\delta \Big|^{-1/2} dx_{p_1} \Big\}^{-2} d\tau = \text{invar}, \tag{46}$$

$$B_{Fi} = \int_{(t)} \Big\{ \int_{(x_p)} |a_d|^{-1/2} dx_{p_1} \Big\}^{-2} d\tau = \text{invar}. \tag{47}$$

From (37) and (38), taking into account that α depends only on X, we find the boundary integral invariants

$$B_{FIIT} = l \{ \int q_{T\Theta}^{-1} dX_{4T\Theta} \}^{-1} = \text{invar}, \tag{48}$$

$$B_{FII\Theta} = l \{ \int q_{\Theta T}^{-1} dX_{\Theta T} \}^{-1} = \text{invar}, \tag{49}$$

or, for example,

$$B_{FIIIT} = l \{ \int \alpha_F^{-1} \{ X_{4T\Theta_m} - X_{4T\Theta_F} \}^{-1} dX_{4T\Theta} \}^{-1} = \text{invar}, \tag{50}$$

$$B_{FIII\Theta} = l \{ \int \alpha_{d_F}^{-1} \{ X_{\Theta T_F} - X_{\Theta T_m} \}^{-1} dX_{\Theta T} \}^{-1} = \text{invar}, \tag{51}$$

i.e., only two invariants for the given type of boundary conditions.

If the quantities $X_{4T\Theta}$, $X_{\Theta T}$, X_{1T}, $X_{1\Theta}$, t, l satisfy a group of similar transformations on the whole, then we find the invariants of this group in the simplest form to be

$$B_{Fo} = \frac{t}{l^2}\,\frac{X_{4T\Theta}}{X_{1T}} = \text{invar},$$
$$B_{Fi} = \frac{t}{l^2}\,\frac{X_{\Theta T}}{X_{1\Theta}} = \text{invar} \qquad (52)$$

and similarly, for example,

$$B_{FIIT} = lq_{T\Theta} X_{4T\Theta}^{-1} = \text{invar},$$

$$B_{FII\Theta} = lq_{\Theta T} X_{\Theta T}^{-1} = \text{invar}. \qquad (53)$$

Thus, for meta-constant phenomena of heat and mass transfer, the system of generalized differential equations and boundary conditions that has been found allows us to describe the phenomena being modeled at the internal points $X_L \in L$ by only two integral invariants B_{Fo}, B_{Fi}, and at the boundary of the surface FL, by the other two integral invariants B_{FT}, $B_{F\Theta}$, or by the four invariants B_{FT1}, $B_{F\Theta}$, B_{FT2}, $B_{F\Theta 2}$ when there are two boundary surfaces (for example, with two parallel planes). Therefore, the total number of integral invariants here is 2 + 2 = 4, or 2 + 4 = 6, instead of 4 + 4 = 8, or 4 + 8 = 12 invariants, required for the primary system (26)–(30). The assembly of invariants derived may serve as a basis for the corresponding empirical investigations in specific problems of metaconstant phenomena of heat and mass transfer.

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